QUAD PRIZE SUBMISSION
SIMULATING ELEMENTARY CAS WITH TRID CAS

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ABSTRACT. The *Trid neighbourhood* is a three cell neighbourhood on a 2-dimensional lattice, and is a subset of the Quad neighbourhood. I show that, with an appropriately prepared initial configuration of "diagonal stripes", a given elementary CA can be simulated by the two state Trid neighbourhood CA whose local update rule is precisely that of the elementary CA. Thus, invoking Cook's universality result for elementary CA rule 110, I exhibit a universal two state Trid neighbourhood CA.

1. Definitions and notation

The *Trid neighbourhood* on a 2-dimensional square lattice is defined by the neighbour offsets

$$\langle (0,1), (0,0), (1,0) \rangle \ . \tag{1}$$

In other words, the neighbourhood of a given cell consists of the cell itself, and the cells immediately adjacent to the north and to the east. A *Trid CA* is a CA with two states (i.e. with state set $\mathbb{Z}_2 = \{0,1\}$) using the Trid neighbourhood. A local update rule for a Trid CA is a function $\mathbb{Z}_2^3 \to \mathbb{Z}_2$.

An *elementary cellular automaton (ECA)* is a 1-dimensional CA with two states and the three cell neighbourhood

$$\langle -1, 0, 1 \rangle \ . \tag{2}$$

Thus a local update rule for an ECA is also a function $\mathbb{Z}_2^3 \to \mathbb{Z}_2$.

A *configuration* of a CA is a mapping of a state to each cell in the lattice. Thus for a D-dimensional two state CA, a configuration is a function $\mathbb{Z}^D \to \mathbb{Z}_2$. The *global map* for a CA is the extension of its local update rule to a function from configurations to configurations, in the usual way.

2. Simulating ECAs with Trid CAs

First, define a mapping α from 1-D configurations to 2-D configurations. Let $s : \mathbb{Z} \to \mathbb{Z}_2$ be a 1-D configuration. Then the 2-D configuration $\alpha(s)$ is defined by

$$\alpha(s)[x,y] = s[x-y] \quad \text{for all } x, y \in \mathbb{Z} \ . \tag{3}$$

Intuitively, $\alpha(s)$ is obtained by mapping the cell states of s along the diagonals which run southwest to northeast. This has the effect that the Trid neighbourhood can "see" the states of three adjacent ECA cells, as depicted in Figure 1.

The mapping α is injective, but not surjective; thus α does not have an inverse. However, define a mapping β from 2-D configurations to 1-D configurations by

$$\beta(s_2)[x] = s_2[x, 0] \quad \text{for all } x \in \mathbb{Z} \tag{4}$$

for all 2-D configurations $s_2 : \mathbb{Z}^2 \to \mathbb{Z}_2$. Then it is easy to see that $\beta \circ \alpha$ is the identity on the set of 1-D configurations. Furthermore, $\alpha \circ \beta$ is the identity on

Date: May 14, 2008.

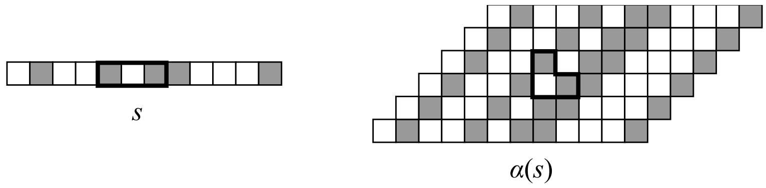

FIGURE 1. Mapping from a 1-D configuration s to the 2-D configuration $\alpha(s)$ defined in Equation 3. The heavy outlines show an instance of the ECA neighbourhood and a corresponding instance of the Trid neighbourhood; the cell states "seen" by these two neighbourhoods are the same.

the subset of 2-D configurations which are constant along each of the southwest–northeast diagonals. This subset is precisely the range of α, and is closed under operation of any Trid CA.

The following theorem is the main result:

Theorem 1. *Let $f : \mathbb{Z}_2^3 \to \mathbb{Z}_2$. Let F_E be the global map obtained by interpreting f as a local update rule for an ECA; similarly let F_T be the global map for f as a Trid CA rule. Then*

$$F_E = \beta \circ F_T \circ \alpha \ . \tag{5}$$

From the discussion of $\alpha \circ \beta$ above, it follows that

$$F_E^k = \beta \circ F_T^k \circ \alpha \tag{6}$$

for all k. This result says that, for any given ECA, we can find a Trid CA which "simulates" the ECA, in the sense that, modulo the appropriate conversions α and β between 1-D and 2-D configurations, the two CAs yield precisely the same evolution from a given initial configuration.

Proof of Theorem 1. Let s be a 1-D configuration, and let $x \in \mathbb{Z}$. It suffices to show that

$$F_E(s)[x] = (\beta \circ F_T \circ \alpha)(s)[x] \ . \tag{7}$$

The right hand side gives

$$\begin{aligned}
\beta(F_T(\alpha(s)))[x] &= F_T(\alpha(s))[x,0] && \text{by Equation 4} \quad (8)\\
&= f\left(\alpha(s)[x,1], \alpha(s)[x,0], \alpha(s)[x+1,0]\right) && \text{by def. of } F_T \quad (9)\\
&= f\left(s[x-1], s[x], s[x+1]\right) && \text{by Equation 3} \quad (10)\\
&= F_E(s)[x] && \text{by def. of } F_E\ , \quad (11)
\end{aligned}$$

as required. □

3. UNIVERSALITY

Cook [1] exhibits a universal ECA (*rule 110* according to Wolfram's numbering scheme [2]). Clearly then, the Trid CA which simulates rule 110 is also universal.

REFERENCES

[1] Matthew Cook. Universality in elementary cellular automata. *Complex Systems*, 15(1):1–40, 2004.

[2] Stephen Wolfram. Statistical mechanics of cellular automata. *Reviews of Modern Physics*, 55(3):601–644, Jul 1983.

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