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The Inverse Behavior of a Reversible One-Dimensional Cellular Automaton Obtained by a Single Welch Diagram

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Reversible cellular automata are discrete dynamical systems based on local interactions which are able to produce an invertible global behavior. Reversible automata have been carefully analyzed by means of graph and matrix tools, in particular the extensions of the ancestors in these systems have a complete representation by Welch diagrams. This paper illustrates how the whole information of a reversible one-dimensional cellular automaton is conserved at both sides of the ancestors for sequences with an adequate length. We give this result implementing a procedure to obtain the inverse behavior by means of calculating and studying a single Welch diagram corresponding with the extensions of only one side of the ancestors. This work is a continuation of our study about reversible automata both in the local [15] and global [16] sense. An illustrative example is also presented.

Keywords: Cellular automata, reversibility, welch indices.

1 INTRODUCTION

Cellular automata are discrete dynamical systems which are able to yield complex behaviours by means of simple interactions. The concept began with John von Neumann and his work on self-reproducing systems which can be consulted in reference [20]. Other relevant works in this field can be consulted in reference [1] and [23].

Reversible cellular automata are a special case of cellular automata where the global mapping is invertible, that is, every global state of the automaton has one and only one successor and the dynamics is deterministic in both directions of time [13]. Reversible one-dimensional cellular automata have been used for modeling and understanding reversible physical and chemical phenomena [18, 22], as well as for implementing data coding systems [3, 19, 22]. The study of reversible automata was first treated in references [8] and [10], and the main reference for the one-dimensional case is provided in [4] studying reversible automata as automorphisms of the full shift, presenting the topological properties of these systems. Another important work about reversible one-dimensional cellular automata and their graph presentations is provided in [11] and [12]. These papers have inspired other studies about reversible automata based on graph presentations: [6, 7, 17], and matrix tools: [9, 21].

One of the problems about reversible one-dimensional cellular automata is how the information of the system is conserved and how we can use the same to find the inverse behavior, in this sense there is a procedure provided by Nasu using both Welch diagrams associated with the extensions of the ancestors at both sides for obtaining the inverse local rule of a reversible automaton [11]. In this paper we treat and resolve this question showing that we can select any side of the ancestors in a reversible automaton and get its information using a single Welch diagram corresponding with the extensions of the ancestors at this side. In this sense we establish a symbolic and matrix approach for calculating the Welch diagram and obtaining the inverse local rule.

The results described in this work represent an extension of our study for implementing a set of computational procedures which can be used for illustrating specific properties of a given reversible cellular automaton, for instance to obtain the properties of connectivity matrices [15], for calculating the features of the transitive behavior in these systems [16] and now for describing how the same information can be obtained in any side of the ancestors.

The paper is organized as follows: Section 2 provides the basic definitions of one-dimensional cellular automata; it also shows a procedure for transforming any one-dimensional cellular automaton into another of neighborhood size 2 to simplify the analysis. The properties of reversible one-dimensional cellular automata are presented as well. Section 3 describes how the extensions of the ancestors in a reversible automaton are represented by Welch diagrams and the properties of such diagrams. Section 4 develops matrix procedures for obtaining the right Welch diagram associated with a particular reversible automaton, these procedures are also useful to get important features of this diagram and for calculating the inverse local rule of the automaton by means of symbolic matrix products. Section 5 illustrates the previous results using a reversible automaton of 4 states. Finally, Section 6 provides the concluding remarks of the paper.

2 PROPERTIES OF REVERSIBLE ONE-DIMENSIONAL CELLULAR AUTOMATA

A one-dimensional cellular automaton consists of a one-dimensional array of cells where each cell initially takes a single state from a finite set K; the initial array of states is the initial configuration of the automaton. Let k be the cardinality of K and for $n \in \mathbb{Z}^+$, let K^n be the set of words with n states. Let K^* be the whole set of finite words and for $w \in K^*$, let w^* be the word formed by the undefined (but finite) repetition of w. For $m, n \in \mathbb{Z}^+$, $w \in K^m$ and $v \in K^n$, $wv \in K^{m+n}$ is an extension of w of length n, and for $w, v \in K^*$, $wv \in K^*$ is just a finite extension of w.

The dynamics of the cellular automaton is defined by local interactions of the cells in the initial array. For $m \in \mathbb{Z}^+$ there is a mapping $\varphi : K^m \to K$ where each $w \in K^m$ is a neighborhood, m is the size of the neighborhood and φ is the local rule of the automaton. Every neighborhood yields a single state of K and φ is applied over each neighborhood in the initial configuration, where every neighborhood shares $m-1$ cells with the contiguous neighborhoods at both sides. In this way the initial configuration produces a new one by the action of the local rule φ, and the global behavior of the automaton depends on the properties of this rule. Let $\mathcal{A} = (k, m, \varphi)$ represent a one-dimensional cellular automaton of k states, neighborhood size m and local rule φ.

Take a cellular automaton $\mathcal{A} = (k, m, \varphi)$, for $a \in K$ and $w \in K^m$, if $\varphi(w) = a$ then w is an ancestor of a while a is the descendant of w. We can note that w has $m-1$ more states than a. We extend this concept for larger words, for $n \in \mathbb{Z}^+$, $n \geq m$ and $v \in K^n$, let $\varphi(v)$ be word yielded by the local rule φ applied over each one of the $n-m+1$ overlapping neighborhoods forming v, hence $\varphi(v) = w \in K^{n-m+1}$ and v is an ancestor of w.

We can use this property for transforming any cellular automaton $\mathcal{A} = (k, m, \varphi)$ into a new cellular automaton $\mathcal{A}' = (k^{m-1}, 2, \tau)$. This transformation was independently explained by Boykett and Kari; a description of this process can be consulted in [2] and [5]. The relevance of this result is that we need to study only cellular automata of the type $\mathcal{A} = (k, 2, \varphi)$ for understanding all the other cases, hence in the rest of this paper we shall only treat these automata. In this case the local rule φ is represented by a matrix M_φ where the row and column indices are the elements of K and the entry $(i, j) = a$ in M_φ if $\varphi(i, j) = a \in K$.

TABLE 1
Ordered pairs of states defining the nodes of the pair diagram.

(0,0)	(0,1)	(0,2)	$\cdots$	(0,$k-1$)
(1,0)	(1,1)	$\cdots$		$\vdots$
$\vdots$	$\vdots$	(2,2)		$\vdots$
$\vdots$			$\ddots$	$\vdots$
($k-1$,0)	$\cdots$	$\cdots$	$\cdots$	($k-1$,$k-1$)

A cellular automaton $\mathcal{A} = (k, 2, \varphi)$ is reversible if there exists a local rule φ^{-1} (possibly with neighborhood size $m \neq 2$) such that it makes invertible the global behavior of $\mathcal{A}$. Reversible automata have been widely studied by their theoretical relevance and their practical applications, one of the most detailed works being developed in [4] using a topological and a combinatorial approach. In particular, Hedlund proves two important properties of these systems; let $\mathcal{A} = (k, 2, \varphi)$ be a reversible automaton and let φ^{-1} be its inverse local rule of neighborhood size m, then $\mathcal{A}$ has the following properties:

Property 1 (Uniform multiplicity of ancestors) *Every word $w \in K^*$ has k ancestors.*

Property 2 (Welch indices) *For $n \geq m$, the ancestors of every word $w \in K^n$ have L possible states in the lefmost position, converge into a unique state and from this one, the ancestors have R possible states in the rightmost position fulfilling that $LR = k$.*

The value L is the left Welch index and R is the right Welch index of the automaton, thus there is a unique way in which every word $w \in K^n$ returns in the evolution of the automaton, and its ancestors have L initial states, a common central part and R final states. One way of knowing if a cellular automaton $\mathcal{A} = (k, 2, \varphi)$ is reversible is constructing its pair diagram. In this diagram, nodes are all the ordered pairs of states, these nodes can be arranged as Table 1 indicates.

For two ordered pairs (a, b) and (a', b'), there is directed edge from (a, b) to (a', b') if $\varphi(a, a') = \varphi(b, b')$. We can detect if the automaton $\mathcal{A}$ is reversible reviewing the cycles of the pair diagram. If there exists a cycle of length m formed by ordered pairs outside of the main diagonal in Table 1, then this cycle contains an ordered pair (a, b) with $a \neq b$. Hence for $a, b \in K$ and $u, v \in K^{m-1}$, there is a word $w \in K^m$ with two ancestors; one with form $aua \in K^{m+1}$ and another with form $bvb \in K^{m+1}$. Thus for any $n \in \mathbb{Z}^+$, the word w^n formed by n repetitions of w has two possible ancestors, $((au)^n)a$ and $((bv)^n)b$; but this implies that the automaton cannot be reversible. In this way, $\mathcal{A}$ is reversible if the cycles of the pair diagram are only formed by the nodes from the main diagonal of Table 1.

For a reversible automaton $\mathcal{A} = (k, 2, \varphi)$, let $m \in \mathbb{Z}^+$ be the minimum length such that the ancestors of each $w \in K^m$ have Welch indices $LR = k$. Different words in K^m have different ancestors, thus for the ancestors of every word in K^m, the set of L initial states defines a left Welch subset and the set of R final states establishes a right Welch subset.

Another relevant paper about reversible cellular automata is presented by Nasu applying graph theory [11]. In particular, Nasu proves the following property:

Property 3 (Intersection property) *For a reversible automaton $\mathcal{A} = (k, 2, \varphi)$, every left Welch subset has one and only one common state with any right Welch subset.*

Property 3 defines a unique way to return in the evolution of the reversible automaton when we take finite configurations with periodic boundary conditions, this will be illustrated in the example of Section 5.

Nasu defines two graphs using Welch subsets, the first is the left Welch diagram formed by all the left Welch subsets and the second is the right Welch diagram composed by all the right Welch subsets. In the next section we briefly explain these definitions, a complete exposition of them can be consulted in [11]. Nasu also provides a complete characterization for Welch diagrams; based on these results we shall prove that a single Welch diagram is enough to obtain the inverse rule of a reversible automaton.

3 WELCH DIAGRAMS

Let m be the minimum length such that the ancestors of every $w \in K^m$ have L initial states, R final states fulfilling that $LR = k$ and a central common state. For the ancestors of every $w \in K^m$, the set of L initial states defines a left Welch subset $W_L \subseteq K$ and the set of R final states specifies a right Welch subset $W_R \subseteq K$. Take the whole set of left Welch subsets associated with a reversible automaton $\mathcal{A} = (k, 2, \varphi)$, with them we shall define a new diagram as follows:

- The nodes of the diagram are the left Welch subsets in $\mathcal{A}$.
- For two nodes W_{L_1} and W_{L_2}, there is a directed link labeled by $a \in K$ going from W_{L_1} to W_{L_2} if for each element $j \in W_{L_2}$ there is another element $i \in W_{L_1}$ such that $(j, i) = a$ in M_φ.

The previous diagram is the left Welch diagram $\mathbf{W}_L$ of the reversible automaton $\mathcal{A}$; in a similar way the right Welch diagram $\mathbf{W}_R$ is defined:

- The nodes of $\mathbf{W}_R$ are the right Welch subsets in $\mathcal{A}$.
- For two nodes W_{R_1} and W_{R_2}, there is a directed link labeled by $a \in K$ going from W_{R_1} to W_{R_2} if for each element $j \in W_{R_2}$ there is another element $i \in W_{R_1}$ such that the entry $(i, j) = a$ in M_φ.

Welch diagrams provide a graph representation for the extensions associated with the ancestors of a given word. Using the theory of definite automata [14], Nasu proves four main properties of these diagrams:

Property 4 (Well-defined diagram) *For every $a \in K$, every node W_L in $\mathbf{W}_L$ has a single outgoing edge labeled by a. This is analogous for $\mathbf{W}_R$.*

Property 5 (Strongly-connected diagram) *$\mathbf{W}_L$ is strongly connected. This is analogous for $\mathbf{W}_R$.*

Property 6 (Mergible diagram) *There exists $p \in \mathbb{Z}^+$ such that, for all $n \geq p$, each path of length n in $\mathbf{W}_L$ begins from a unique state of the initial left Welch subset. In this case $\mathbf{W}_L$ is p-mergible, this is analogous for $\mathbf{W}_R$.*

Property 7 (Definite diagram) *There exists $q \in \mathbb{Z}^+$ such that, for all $n \geq q$ and all $w \in K^n$, all the paths labeled by w in $\mathbf{W}_L$ have a single final left Welch subset. Thus $\mathbf{W}_L$ is q-definite, this is analogous for $\mathbf{W}_R$.*

In the following section we shall present a set of procedures for detecting the values p and q for a reversible automaton such that the right Welch diagram is p-mergible and q-definite. These procedures are based on the matrix M_φ and p, q will be used for specifying another process for calculating the inverse rule φ^{-1} by means of the right Welch diagram. Therefore the next results just discuss right Welch diagrams, but they are also analogous for left Welch diagrams. For simplicity, in the following sections every right Welch subset is represented by W and the right Welch diagram is referred by $\mathbf{W}$.

4 PROCEDURES FOR OBTAINING THE INVERSE LOCAL RULE

For a reversible automaton $\mathcal{A} = (k, 2, \varphi)$, the right Welch diagram $\mathbf{W}$ has a matrix representation M_W where the row and column indices are the nodes of the diagram, and every entry (i, j) in M_W shows the label of the edges from node i to node j in $\mathbf{W}$; the entry may have no element, one, or several elements depending on the edges joining both nodes. Let us define a procedure for obtaining the matrix M_W using M_φ.

Procedure 1 (Transition from M_φ to M_W)

1 *For every row a in M_φ and every state $b \in K$, take the subset $C \subseteq K$ such that for each $c \in C$ the entry $(a, c) = b$ in M_φ. The distinct subsets C_i are indexed by $i \in \mathbb{Z}^+$.*
2 *Form a new matrix A_0 where the row indices are the elements of K and the column indices are the subsets C_i formed in the previous step. Each entry in A_0 has the form (a, C_i) and $(a, C_i) = b$ if $(a, c) = b$ in M_φ for all $c \in C_i$. Otherwise, there is no element in the entry (a, C_i).*

3 *For every subset C_i and every state $b \in K$, take the subset $D \subseteq K$ such that for all $d \in D$, there exists $c \in C_i$ such that $(c, d) = b$ in M_φ. The distinct subsets D_j are indexed by $j \in \mathbb{Z}^+$.*

4 *Form a new matrix A_1 where the row indices are the subsets C_i and the column indices are the subsets D_j formed in the previous step. Each entry in A_1 has the form (C_i, D_j) and $(C_i, D_j) = b$ if $(c, d) = b$ in M_φ for $c \in C_i$ and $d \in D_j$. Otherwise there is no element in the entry (C_i, D_j).*

5 *If the subsets C_i are equal to the subsets D_j then stop the procedure and $A_1 = M_W$, otherwise repeat Step 3 for the subsets D_j to form a new matrix A_2.*

With Procedure 1 we can prove the following result:

Theorem 1 *For a reversible automaton $\mathcal{A} = (k, 2, \varphi)$, if Procedure 1 yields the sequence of matrices $A_0 \ldots A_{p+1}$ for $p \in \mathbb{Z}^+$ such that $A_{p+1} = M_W$, then* **W** *is p-mergible.*

Proof: The previous procedure keeps the right extensions of the ancestors as subsets of states. These extensions form a sequence of matrices $A_0 \ldots A_{p+1}$ for $p \in \mathbb{Z}^+$, where $M_W = A_{p+1}$. The sequence $A_0 \ldots A_{p+1}$ can be used to show how a particular state is connected with a given right Welch subset; if the entry $(a, C_i) = b$ in A_0 and the entry $(C_i, D_j) = c$ in A_1, then some of the ancestors of the word $bc \in K^2$ begin from $a \in K$ and finish with the states in the subset D_j. We can represent these ancestors by the symbolic product $A_0 A_1$, where "symbolic" means that the product $(a, C_i)(C_i, D_j) = bc$ and the entry (a, D_j) in $A_0 A_1$ is equal to the whole list of distinct products $(a, C_i)(C_i, Dj)$ for all the subsets C_i.

The symbolic product $P = A_0 A_1 \cdots A_{p-1} A_p$ yields a new matrix where the row indices are the states of K and the column indices are the right Welch subsets. Each entry in P has the form $(a, W) = B \subset K^p$, where B is the set of words in K^p whose ancestors begin from $a \in K$ and finish in the right Welch subset $W \subset K$. The same happens for all the words in K^p.

Suppose that there exists $a, b \in K$ and $u \in K^p$ such that for the same right Welch subset W_1, we have that $(a, W_1) = B_1$, $(b, W_1) = B_2$, $B_1 \cap B_2 = u$ and a, b belong to the same right Welch subset W_2. By Property 5, there exists another finite path labeled by $v \in K^*$ in **W** going from W_1 to W_2, in this way the path labeled by $uv \in K^*$ goes from $a, b \in W_2$ to the whole set W_2, hence the path labeled by $(uv)^*$ goes from $a, b \in W_2$ to the whole set W_2 any finite number of times, contradicting Property 6.

Therefore the automaton is p-mergible and every path of length p in **W** starts from a single state for each right Welch subset. □

Theorem 1 provides a procedure for detecting $p \in \mathbb{Z}^+$ such that the reversible automaton $\mathcal{A}$ is p-mergible. In the following, we shall explain how the right Welch diagram is useful to know $q \in \mathbb{Z}^+$ such that the same automaton is also q-definite. For each state $a \in K$, take from M_W a new matrix M_a where the row and column indices are the ones of M_W and each entry (i, j) in M_a is defined as follows:

$$(i, j) = \begin{cases} 1 \text{ if } (i, j) = a \text{ in } M_W \\ \\ 0 \quad \text{ in other case} \end{cases} \tag{1}$$

For $a \in K$ each M_a is the connectivity matrix of a, hence we initially have k connectivity matrices specified by M_W, one for each state of the automaton. We can define connectivity matrices for larger words, for $v \in K^*$ the matrix M_v is produced by the matrix product of the connectivity matrices corresponding with the states forming v. For instance if $a, b, c \in K$ then $M_{abc} = M_a M_b M_c$. Using connectivity matrices, we shall prove the following result.

Theorem 2 *For a reversible automaton $\mathcal{A} = (k, 2, \varphi)$ with a right Welch diagram represented by M_W, if there exists $q \in \mathbb{Z}^+$ such that for each word $v \in K^q$ we have that M_v has a single non-zero column where each entry is 1, then the automaton is q-definite.*

Proof: Each matrix M_a represents the paths in **W** labeled by $a \in K$. If there exists $q \in \mathbb{Z}^+$ such that M_v has a single non-zero column with entries equal to 1 for each $v \in K^q$, then all the paths in **W** labeled by v begin from all the nodes and they end into a unique node of **W**. Therefore all the paths of length q with the same label converge into the same node and the automaton is q-definite. □

Theorem 2 gives a procedure based on matrix products for detecting $q \in \mathbb{Z}^+$ such that the automaton is q-definite.

Procedure 2 (Level q-definite)

1. *Take $q = 1$.*
2. *Take $n = q$ and for each word $v \in K^n$, form the connectivity matrix M_v.*
3. *If each connectivity matrix M_v has a single non-zero column with each entry equal to 1, then the automaton is q-definite and stop the procedure. Otherwise, repeat step 2 taking now $q = n + 1$.*

With these results, we can define a procedure for obtaining the inverse local rule of the automaton. This procedure is based on the matrices used to calculate M_W from M_φ. Let $A_0 \ldots A_{p+1}$ be the sequence of matrices defining the transition from M_φ into M_W. For $0 \le m \le p$, the column indices of the matrix A_m are equal to the row indices of the matrix A_{m+1}, so we can get the symbolic product $A_m A_{m+1}$ which forms a matrix with the row indices of A_m and the column indices of A_{m+1}. Each entry (i, j) in $A_m A_{m+1}$ is formed by all the words yielded by the concatenation of the words in each entry of row i in A_m with all the words in the corresponding entry in column j of A_{m+1}. Thus each entry in $A_m A_{m+1}$ may have several words.

The symbolic product $A_m A_{m+1}$ produces all the words whose ancestors begin from the subsets of states represented by the row indices of A_m to the subsets of states presented by the column indices of A_{m+1}. Therefore each entry in the symbolic product $P = A_0 A_1 \cdots A_p$ shows the words (which are also labeled paths in **W**) going from particular states to right Welch subsets. Using these symbolic products, we define the following procedure for getting the inverse rule of the automaton:

Procedure 3 (Inverse local rule)

1 *Apply Procedures 1 and 2 to obtain p and q respectively such that the automaton is p-mergible and q-definite. Take the value $m = \max\{p, q\}$.*
2 *Obtain the symbolic product $N = A_0 A_1 \cdots A_p$, if $q > p$ then take $n = q - p$ and calculate also the symbolic product $P = N B_1 B_2 \cdots B_n$ where $B_i = A_{p+1}$ for $1 \leq i \leq n$; in other case $P = N$.*
3 *For each word $w \in K^m$, take all the entries in P containing w; by Properties 2 and 7, w appears in L rows and in a single column.*
4 *For $w, v \in K^m$, take the right Welch subset representing the column in P containing w and take the rows in P containing v, these rows form a subset of states which is a left Welch subset.*
5 *By Property 3, the right Welch subset of w and the left Welch subset of v have a single common state $a \in K$; thus for $wv \in K^{2m}$, $\varphi^{-1}(wv) = a$.*

With the previous procedure we obtain the inverse local rule φ^{-1} for a reversible automaton, where the size of the inverse neighborhood is $2m$ and with a inverse centered evolution in every neighborhood.

Procedure 1 takes each state with every evolution in order to obtain the subsets C_i, hence the first iteration has complexity k^2. The second iteration takes all the subsets C_i with every right extension, because every C_i may have several elements and there are at most k^2 subsets, this iteration has complexity k^4. Thus using the same analysis, iteration n has complexity k^{2n} and if the automaton is q-mergible, the procedure will have a complexity of k^{2q} in the last iteration. Since q is in relation with k, then the procedure is exponential. The other procedures depend of Procedure 1, therefore all the procedures have exponential complexity with regard of the number of states.

Although this way of calculating the inverse local rule for a reversible automaton is exponential, this procedure demonstrates that the necessary information for obtaining the inverse behavior of the automaton is in any

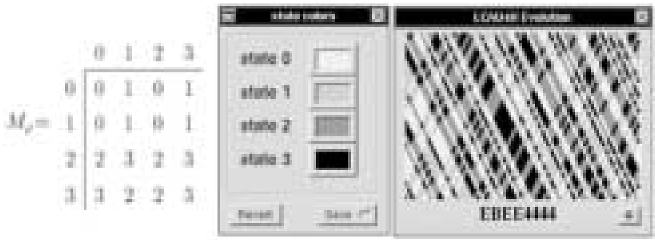

FIGURE 1
Matrix M_φ and one evolution of the automaton $\mathcal{A} = (4, 2, EBEE4444)$.

TABLE 2
Matrix M_W obtained by Procedure 1.

$A_0 =$

	(0,2)	(1,3)	(0,3)	(1,2)
0	0	1		
1	0	1		
2	2	3		
3			3	2

$A_1 =$

	(0,2)	(1,3)	(0,3)	(1,2)
(0,2)	0,2	1,3		
(1,3)	0	1	3	2
(0,3)	0	1	3	2
(1,2)	0,2	1,3		

of the ending parts of the ancestors, whether we choose the right side or the left one, because we can specify analogous procedures for the left Welch diagram.

5 ILLUSTRATIVE EXAMPLE

We shall illustrate the previous results using a reversible automaton $\mathcal{A} = (4, 2, \varphi)$. We choose this example because this is not a trivial one since the inverse local rule is larger than the original one, but its small size allows to obtain a suitable presentation in the paper.

This type of automaton can be identified by a particular number base 16; take the descendants from the neighborhoods from the neighborhoods 33 and 32, suppose that $\varphi(33) = a$ and $\varphi(32) = b$. Take now $x = (a * 4) + b$, then x identifies the evolution of 33 and 32, continuing with the next neighborhoods we have other 7 pairs of descendants, and for each pair we can assign a number x base 16. Thus a sequence of 8 hexadecimal digits identifies the local rule φ. In the example of Figure 1 we have that $\varphi(33) = 3$ and $\varphi(32) = 2$, therefore $((4 * 3) + 2) = 14$ or E base 16.

This corresponds to automaton $\mathcal{A} = (4, 2, EBEE4444)$, which rule's matrix representation and an example of evolution are shown in Figure 1. The pair diagram and its cycles are presented in Figure 2. We can see that the only cycle is composed by the diagonal elements, therefore the automaton is reversible.

Using Procedure 1, we obtain the matrix M_W from M_φ in Table 2.

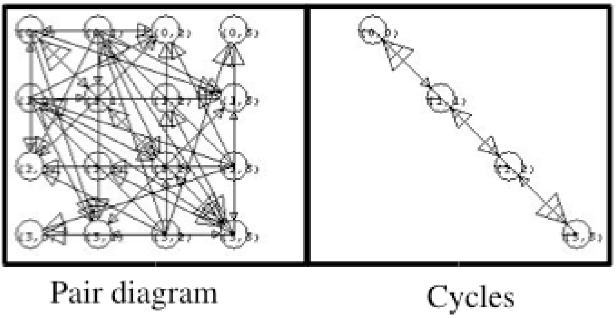

FIGURE 2
Cycles of the pair diagram for the automaton $\mathcal{A} = (4, 2, EBEE4444)$.

TABLE 3
Connectivity matrices obtained from M_W.

$M_0 =$

	0	1	2	3
0	1	0	0	0
1	1	0	0	0
2	1	0	0	0
3	1	0	0	0

$M_1 =$

	0	1	2	3
0	0	1	0	0
1	0	1	0	0
2	0	1	0	0
3	0	1	0	0

$M_2 =$

	0	1	2	3
0	1	0	0	0
1	0	0	0	1
2	0	0	0	1
3	1	0	0	0

$M_3 =$

	0	1	2	3
0	0	1	0	0
1	0	0	1	0
2	0	0	1	0
3	0	1	0	0

The procedure needs two steps to yield M_W, therefore the automaton is 1-mergible by Theorem 1. From the matrix M_W we shall obtain the connectivity matrix of each state (Table 3).

Matrices M_2 and M_3 do not show the expected form described in Theorem 2, hence we shall apply Procedure 2 to get the desired connectivity matrices. For this reason, we calculate the products of the matrices in Table 3 to yield the connectivity matrices of larger words. There are four distinct types of connectivity matrices for the words in K^2, these matrices are presented in Table 4.

Procedure 2 shows that the automaton is 2-definite. We shall apply now Procedure 3 for obtaining the inverse local rule. Since the automaton is 1-mergible and 2-definite, we have to calculate the symbolic product $P = A_0 A_1$ which is presented in Table 5.

TABLE 4
Connectivity matrices M_w for $w \in K^2$.

Distinct connectivity matrices

$$B_0 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \end{array} \quad B_1 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \end{array} \quad B_2 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 \end{array} \quad B_3 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{llll} M_{00} = B_0 & M_{01} = B_1 & M_{02} = B_0 & M_{03} = B_1 \\ M_{10} = B_0 & M_{11} = B_1 & M_{12} = B_3 & M_{13} = B_2 \\ M_{20} = B_0 & M_{21} = B_1 & M_{22} = B_0 & M_{23} = B_1 \\ M_{30} = B_0 & M_{31} = B_1 & M_{32} = B_3 & M_{33} = B_2 \end{array}$$

TABLE 5
Symbolic product $P = A_0 A_1$.

	(0,2)	(1,3)	(0,3)	(1,2)
0	00 02 10	01 03 11	13	12
1	00 02 10	01 03 11	13	12
2	20 22 30	21 23 31	33	32
3	30 20 22	31 21 23	33	32

From Table 5 we can finally obtain the inverse local rule following steps 4 and 5 in Procedure 3, for this example the inverse local rule shall be represented by a matrix (Table 6), where the row and column indices are the words in K^2 and each entry $(w, v) = a \in K$ in this matrix means that $\varphi^{-1}(wv) = a$.

An example of the invertible evolution with a finite configuration for this automaton is given in Figure 3.

TABLE 6
Inverse local rule for the automaton $\mathcal{A} = (4, 2, EBEE4444)$.

	00	01	02	03	10	11	12	13	20	21	22	23	30	31	32	33
00	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2
01	1	1	1	1	1	1	1	1	3	3	3	3	3	3	3	3
02	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2
03	1	1	1	1	1	1	1	1	3	3	3	3	3	3	3	3
10	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2
11	1	1	1	1	1	1	1	1	3	3	3	3	3	3	3	3
12	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
13	0	0	0	0	0	0	0	0	3	3	3	3	3	3	3	3
20	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2
21	1	1	1	1	1	1	1	1	3	3	3	3	3	3	3	3
22	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2
23	1	1	1	1	1	1	1	1	3	3	3	3	3	3	3	3
30	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2
31	1	1	1	1	1	1	1	1	3	3	3	3	3	3	3	3
32	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
33	0	0	0	0	0	0	0	0	3	3	3	3	3	3	3	3

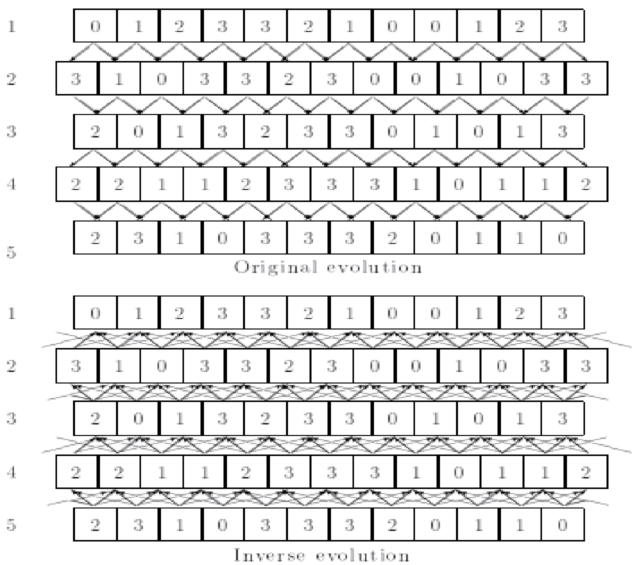

FIGURE 3
Evolution of $\mathcal{A} = (4, 2, EBEE4444)$. The terminal cells in configurations 2 and 4 at both sides are the same cell, we have duplicated these cells just for clarity.

6 CONCLUDING REMARKS

The graph viewpoint and the matrix presentation have been useful for analyzing and providing a set of computable procedures which calculate and characterize the Welch diagrams of a reversible cellular automaton. We use these diagrams for obtaining the inverse behavior.

However, these graphs and matrices have a significant size whether the automaton has a small number of states or the inverse local rule is large. Hence, such procedures are useful only for reversible one-dimensional cellular automata with few states.

The main contribution of the procedures presented in this work is that we just need the information at one side of the ancestors in a reversible automaton for getting its inverse behavior. Therefore both sides of the ancestors have all the needed information to produce the inverse local rule.

A further work is to improve these procedures to obtain a polynomial perfomance, and to apply an adaptation of them to other classes of cellular automata, for instance to analyze the surjective case.

Another extension is to use more specific tools and results from graph theory and symbolic dynamics in order to obtain their relation with the theory of reversible cellular automata; with this we may establish deeper properties for characterizing reversible automata.

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